THE NONSTEADY CONTACT PROBLEM OF HEAT CONDUCTION IN COMPOSITE CYLINDRICAL SYSTEMS WITH THERMAL RESISTANCE

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The heat-conduction problem in a three-layer cylindrical system is solved in this article for the case of a complex boundary layer and heat sources distributed over the surface and the volume.

Cylindrical structures are used extensively in various branches of engineering. We have numerous solutions at our disposal for the temperature problem pertaining to cylindrical structures, including those that are composite [1–3]. However, the real processes are described, as a rule, by quantitative relationships so complex that they cannot be reduced to existing schemes.

In this paper we analyze a generalized model which schematizes a large group of thermal problems associated with the production operations of stamping, extrusion, etc. [4]. Let us consider a three-component cylindrical system: blank-boundary layer-multilayer tool (Fig. 1). The study is carried out for the heating stage of the first cycle in hot stamping. In the deformation of a blank ($0 \le r \le R_0$) with the thermal coefficients λ_0 and a_0, as well as an initial temperature $U_0(r, 0) = v = \text{const}$ for $t > 0$, a constant quantity of heat w is uniformly generated per unit time per unit volume. For the purposes of the example, the tool is made of a two-layer material: the first layer ($R_0 \le r \le R_1$) has the thermal coefficients λ_1 and a_1; for the second layer ($R_1 \le r < \infty$) we have λ_2 and a_2. We assume the contact between the layers to be ideal. The initial tool temperature is assumed to be equal to zero: $U_1(r, 0) = U_2(r, 0) = 0$.

The boundary layer is made up of two sublayers, one of which belongs to the heated blank, with the other a part of the tool. The corresponding thermal resistances—taken with consideration of the contact resistance—are denoted $\rho_0 = 1/H_0$ and $\rho_1 = 1/H_1$ (H is the thermal conductivity). The heat of friction q is generated within the boundary layer and in those portions of the blank and stamp that are in contact. However, with an approach that is arbitrary to some extent, we can assume that q is localized between the sublayers

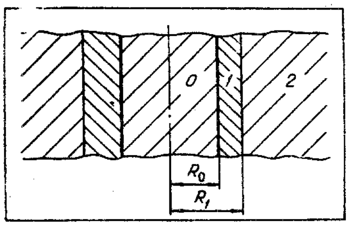

Fig. 1. Scheme of the generalized model.

(Fig. 2). Applying the Kirchhoff and Ohm laws for the "boundary layer" segment of the thermal circuit, we obtain

$$q_0 + q - q_1 = 0,$$
$$U_1(R_0, t) = U_0(R_0, t) - \rho_0 q_0 - \rho_1 q_1, \tag{1}$$

where q_0 is the flow of heat from the blank and q_1 is the heat flow to the stamp:

$$q_0 = -\lambda_0 \frac{\partial}{\partial r} U_0(R_0, t),$$
$$q_1 = -\lambda_1 \frac{\partial}{\partial r} U_1(R_0, t).$$

Then, relying on (1), we will formulate the stated problem as follows:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{a_0}\frac{\partial}{\partial t}\right) U_0(r, t) = -\frac{w}{\lambda_0}$$
$$(0 < r < R_0,\ t > 0),$$
$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{a_j}\frac{\partial}{\partial t}\right) U_j(r, t) = 0$$
$$(j = 1,\ R_0 < r < R_1;\ j = 2,\ R_1 < r < \infty),$$
$$-\lambda_0 \frac{\partial}{\partial r} U_0(R_0, t) + \lambda_1 \frac{\partial}{\partial r} U_1(R_0, t) + q = 0 \quad (t > 0),$$
$$\lambda_0 \frac{\partial}{\partial r} U_0(R_0, t)$$
$$+ H\left[U_0(R_0, t) - U_1(R_0, t) - \frac{q}{H_1}\right] = 0$$
$$(H^{-1} = H_0^{-1} + H_1^{-1}),$$
$$\lambda_1 \frac{\partial}{\partial r} U_1(R_1, t) = \lambda_2 \frac{\partial}{\partial r} U_2(R_1, t),$$
$$U_1(R_1, t) = U_2(R_1, t),$$
$$U_0(0, t) < \infty, \quad U_2(\infty, t) = 0,$$
$$U_0(r, 0) = v = \text{const},$$
$$U_1(r, 0) = U_2(r, 0) = 0. \tag{2}$$

Applying the unilateral Laplace transform with respect to the time t,

$$\bar{U}_j(r, s) = \int_0^\infty U_j(r, t) e^{-st} dt \quad (j = 0, 1, 2),$$

to the boundary problem (2), in the image space we obtain

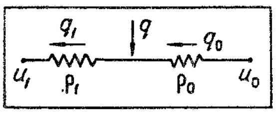

Fig. 2. Scheme of the boundary layer.

$$\bar{U}_0(r,\ s)=\frac{v}{s}+\frac{wa_0}{\lambda_0 s^2}-\frac{H\,[\Delta_2(I)\,\Delta_1(K)-\Delta_2(K)\,\Delta_1(I)]\,I_0(p_0 r)}{s\,[\Delta_0(I)\,\Delta_1(K)-\Delta_0(K)\,\Delta_1(I)]},$$

$$\bar{U}_1(r,\ s)=\frac{\delta\,[\Delta_1(K)\,I_0(p_1 r)-\Delta_1(I)\,K_0(p_1 r)]}{s\,[\Delta_0(I)\,\Delta_1(K)-\Delta_0(K)\,\Delta_1(I)]},$$

$$\bar{U}_2(r,\ s)=\frac{\lambda_1\delta\,K_0(p_2 r)}{sR_1\,[\Delta_0(I)\,\Delta_1(K)-\Delta_0(K)\,\Delta_1(I)]}, \qquad (3)$$

where $p_j=\sqrt{s/a_j}$, and $Z_\nu(x)$ denote the cylindrical functions $I_\nu(x)$ and $K_\nu(x)$—of order ν—of the purely imaginary argument,

$$\Delta_0(Z)=\begin{vmatrix} -p_0\lambda_0 I_1(p_0R_0) & \lambda_1 p_1 Z_0'(p_1R_0) \\ p_0\lambda_0 I_1(p_0R_0)+HI_0(p_0R_0) & -HZ_0(p_1R_0)\end{vmatrix},$$

$$\Delta_1(Z)=\begin{vmatrix} \lambda_1 p_1 Z_0'(p_1R_1) & \lambda_2 p_2 K_1(p_2R_1) \\ Z_0(p_1R_1) & -K_0(p_2R_1)\end{vmatrix},$$

$$\Delta_2(Z)=\begin{vmatrix} -q & \lambda_1 p_1 Z_0'(p_1R_0) \\ \dfrac{q}{H_1}-v-\dfrac{wa_0}{\lambda_0 s} & -Z_0(p_1R_0)\end{vmatrix};$$

$$\delta=\begin{vmatrix} -p_0\lambda_0 I_1(p_0R_0) & -q \\ p_0\lambda_0 I_1(p_0R_0)+HI_0(p_0R_0) & \left(\dfrac{q}{H_1}-v-\dfrac{wa_0}{\lambda_0 s}\right)H\end{vmatrix}.$$

For s = 0 the transforms $\bar{U}_j(r,s)$ have a branch point, and this is easily established [5] by taking the first terms in the series expansion of the following cylindrical functions for the small values of the argument:

$$I_0(x)\simeq 1,\quad I_1(x)\simeq\frac{x}{2},$$

$$K_0(x)\simeq \ln\frac{Cx}{2},\quad K_1(x)\simeq\frac{1}{x}+\frac{x}{2}\ln\frac{Cx}{2}-\frac{x}{4},$$

where $\gamma=\ln C=0.577\ldots$ is the Euler constant.

Therefore, turning to the original with the Riemann–Mellin inversion formula, we should assume an integration contour that is cut along the negative real half-axis. At and within this contour the $\bar{U}_j(r,s)$ are uniquely defined functions of s and have no poles, which is easily proved, using asymptotic expressions for the cylindrical functions when $x\gg\nu$,

$$I_\nu(x)\simeq\frac{1}{\sqrt{2\pi x}}e^x,\quad K_\nu(x)\simeq\sqrt{\frac{\pi}{2x}}e^{-x},$$

if we investigate the behavior of these functions for large values of s:

$$\bar{U}_0(r,\ s)\simeq\frac{v}{s}+\frac{wa_0}{\lambda_0 s^2}+\frac{H\sqrt{\dfrac{R_0}{r}}\,e^{-p_0(R_0-r)}\left[q-\lambda_1 p_1\left(v-\dfrac{q}{H_1}+\dfrac{wa_0}{\lambda_0 s}\right)\right]}{s(\lambda_0\lambda_1 p_0p_1+\lambda_0 p_0 H+\lambda_1 p_1 H)},$$

$$\bar{U}_1(r,\ s)\simeq\sqrt{\frac{R_0}{r}}\,e^{-p_1(r-R_0)}\times\frac{\left[q\lambda_0 p_0+qH+\lambda_0 p_0 H\left(v-\dfrac{q}{H_1}+\dfrac{wa_0}{\lambda_0 s}\right)\right]}{s(\lambda_0\lambda_1 p_0p_1+\lambda_0 p_0 H+\lambda_1 p_1 H)} \qquad (4)$$

$$\bar{U}_2(r,\ s)\simeq\frac{2\dfrac{\lambda_1}{\sqrt{a_1}}}{\dfrac{\lambda_1}{\sqrt{a_1}}+\dfrac{\lambda_2}{\sqrt{a_2}}}\times\sqrt{\frac{R_0}{r}}\exp\left\{p_1\left[R_1-R_0+\sqrt{\frac{a_1}{a_2}}(r-R_1)\right]\right\}\times\frac{q\lambda_0 p_0+qH+\lambda_0 p_0 H\left(v-\dfrac{q}{H_1}+\dfrac{wa_0}{\lambda_0 s}\right)}{s(\lambda_0\lambda_1 p_0p_1+\lambda_0 p_0 H+\lambda_1 p_1 H)}.$$

On the basis of the residue theorem, the integrals with respect to the selected contour are equal to zero, while on the basis of the Jordan lemma the integrals with respect to the great arcs of the integration contour approach zero when the radius of the great circle increases without limit. At the limit, the integrals with respect to the small circle also yield zero. Assuming $s=a_0x^2e^{\pi i}$ at the upper cut line, and $s=a_0x^2e^{-\pi i}$ at the lower cut line, we then obtain *

$$U_0(r,\ t)=\frac{2H}{\pi}\int_0^\infty\frac{\varphi(N)\,\psi(J)-\varphi(J)\,\psi(N)}{\psi^2(J)+\psi^2(N)}\left(1-e^{-a_0x^2t}\right)\frac{dx}{x},$$

$$U_1(r,\ t)=\frac{2}{\pi}\int_0^\infty[\psi(N)\,\gamma(J)-\psi(J)\,\gamma(N)]\,\frac{E\left(1-e^{-a_0x^2t}\right)}{\psi^2(J)+\psi^2(N)}\,\frac{dx}{x},$$

$$U_2(r,\ t)=\frac{4}{\pi^2}\times\int_0^\infty\frac{\psi(J)\,N_0(bxr)-J_0(bxr)\,\psi(N)}{\psi^2(J)+\psi^2(N)}\left(1-e^{-a_0x^2t}\right)\frac{dx}{x}, \qquad (5)$$

where

$$\varphi(Z)=D(J)\,C(Z)+D(N)\,B(Z),$$

$$\psi(Z)=A(J)\,C(Z)+A(N)\,B(Z),$$

$$\gamma(Z)=C(Z)\,J_0(axr)+B(Z)\,N_0(axr);$$

$$A(Z)=x\begin{vmatrix}\lambda_0 J_1(xR_0) & a\lambda_1 Z_1(axR_0)\\ -\lambda_0 xJ_1(xR_0)+HJ_0(xR_0) & HZ_0(axR_0)\end{vmatrix},$$

$$B(Z)=-x\begin{vmatrix}\lambda_1 aJ_1(axR_1) & \lambda_2 bZ_1(bxR_1)\\ J_0(axR_1) & Z_0(bxR_1)\end{vmatrix},$$

$$C(Z)=x\begin{vmatrix}-\lambda_1 aN_1(axR_1) & \lambda_2 bZ_1(bxR_1)\\ N_0(axR_1) & -Z_0(bxR_1)\end{vmatrix},$$

$$D(Z)=\begin{vmatrix}-q_1 & \lambda_1 axZ_1(axR_0)\\ \dfrac{q}{H_1}-v+\dfrac{w}{\lambda_0 x^2} & Z_0(axR_0)\end{vmatrix},$$

$$E=\begin{vmatrix}x\lambda_0 J_1(xR_0) & -q\\ -x\lambda_0 J_1(xR_0)+HJ_0(xR_0) & \left(\dfrac{q}{H_1}-v+\dfrac{w}{\lambda_0 x^2}\right)H\end{vmatrix};$$

$Z_\nu(x)$ are cylindrical functions—of order ν—of the real argument,

$$a=\sqrt{a_0/a_1},\quad b=\sqrt{a_0/a_2}.$$

*The originals are found for $s\bar{U}_j(r,s)$ on each of the cut lines and the results are added. The functions $U_j(r,t)$ are then determined on the basis of the integral-image theorem.

The cited solutions—involving the use of contemporary computer facilities—enable us to find the temperature field for sources of any duration.

Many processes, stamping in particular, are characterized by sources of brief duration. An analysis of such processes would be useful, if we rely on the simpler solution derived for small periods of time, proceeding from the familiar limit theorem of operational calculus. In conformity with (5), as $t \to 0$, we have

$$U_0(r, t) \approx v + \frac{wa_0}{\lambda_0} t + \frac{H\sqrt{\frac{R_0}{r}}}{\lambda_0\lambda_1 a h_0}\varphi(x, t),$$

$$U_1(r, t) \approx H\frac{a\sqrt{\frac{R_0}{r}}}{\lambda_0\lambda_1 h_1}\psi(y, t),$$

$$U_2(r, t) \approx H\frac{a\sqrt{\frac{R_0}{r}}}{\lambda_0\lambda_1 h_1}\,\frac{2\frac{\lambda_1}{\sqrt{a_1}}}{\frac{\lambda_1}{\sqrt{a_1}}+\frac{\lambda_2}{\sqrt{a_2}}}\psi(z, t), \tag{6}$$

where

$$\varphi(x, t) = A_0\left[\exp(h_0x + a_0th_0^2)\operatorname{erfc}\left(\frac{x}{2\sqrt{a_0t}} + h_0\sqrt{a_0t}\right) - \operatorname{erfc}\frac{x}{2\sqrt{a_0t}}\right] + B_0\sqrt{a_0t}\,2i\operatorname{erfc}\left(\frac{x}{2\sqrt{a_0t}}\right) - \frac{\lambda_1 aw}{\lambda_0}a_0t\,4i^2\operatorname{erfc}\left(\frac{x}{2\sqrt{a_0t}}\right),$$

$$\psi(y, t) = A_1\left[\exp(h_1y + a_1th_1^2)\operatorname{erfc}\left(\frac{y}{2\sqrt{a_1t}} + h_1\sqrt{a_1t}\right) - \operatorname{erfc}\left(\frac{y}{2\sqrt{a_1t}}\right)\right] + B_1\sqrt{a_1t}\,2i\operatorname{erfc}\left(\frac{y}{2\sqrt{a_1t}}\right) + aa_1wt\,4i^2\operatorname{erfc}\left(\frac{y}{2\sqrt{a_1t}}\right),$$

$$A_0 = \frac{q}{h_0} + \lambda_1 a\left(v - \frac{q}{H_1} + \frac{w}{\lambda_0 h_0^2}\right),$$

$$B_0 = q + \frac{\lambda_1 aw}{\lambda_0 h_0},$$

$$A_1 = \frac{q}{H_1} - \frac{aw}{h_1^2} - \frac{v + \frac{q}{H_2}}{a}\lambda_0, \quad B_1 = q - \frac{aw}{h_1},$$

$$x = R_0 - r, \quad y = r - R_0,$$

$$z = R_1 - R_0 + \sqrt{\frac{a_1}{a_2}}(r - R_1),$$

$$h_0 = H\left(\frac{1}{\lambda_0} + \frac{1}{a\lambda_1}\right),$$

$$h_1 = H\left(\frac{a}{\lambda_0} + \frac{1}{\lambda_1}\right) = ah_0.$$

When we use (6), it is not difficult to determine the magnitude of the heat flow acting on the surface of the stamping tool ($r = R_0$) during the period of active contact:

$$q_1 \approx \frac{aH}{\lambda_0 h_1}\left(\frac{1}{2R_0}\left\{A_1[\exp(a_1h_1^2t)\operatorname{erfc}(h_1\sqrt{a_1t}) - 1] + 1.1284B_1\sqrt{a_1t} + aa_1wt\right\} - A_1h_1\exp(a_1h_1^2t)\operatorname{erfc}(h_1\sqrt{a_1t}) + B_1 + 1.1284aw\sqrt{a_1t}\right). \tag{7}$$

There is obvious interest in finding q_1 theoretically.

NOTATION

a_i is the thermal diffusivity; λ is the thermal conductivity; r is the instantaneous radius; R_0 is the radius of "blank-tool" joint; R_1 is the joint radius for composite bilayer cylindrical tool; U_0 is the initial temperature of the blank; U_1 and U_2 are the initial temperatures of the cylindrical rings of the composite bilayer tool; q is the specific heat flux due to heat of friction; w is the volumetric output of the heat source due to deformation; q_0 is the power of the heat flux from the blank; q_1 is the power of the heat flux to the stamp; ρ is the thermal resistance; H is the thermal conductivity; $I_0(x)$, $I_1(x)$, $K_0(x)$, and $K_1(x)$ are the cylindrical (Bessel) zero-th and first order functions of pure imaginary argument; ν is the order of the cylindrical function; $2i\operatorname{erfc}(x)$ and $4i^2\operatorname{erfc}(x)$ are special functions.

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